

A new class of hyper-bent Boolean functions in binomial forms

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Abstract Bent functions, which are maximally nonlinear Boolean functions with even numbers of variables and whose Hamming distance to the set of all affine functions equals $2^{n-1} \pm 2^{\frac{n}{2}-1}$, were introduced by Rothaus in 1976 when he considered problems in combinatorics. Bent functions have been extensively studied due to their applications in cryptography, such as S-box, block cipher and stream cipher. Further, they have been applied to coding theory, spread spectrum and combinatorial design. Hyper-bent functions, as a special class of bent functions, were introduced by Youssef and Gong in 2001, which have stronger properties and rarer elements. Many research focus on the construction of bent and hyper-bent functions. In this paper, we consider functions defined over \mathbb{F}_{2^n} by $f_{a,b} := \text{Tr}_1^n(ax^{(2^m-1)}) + \text{Tr}_1^4(bx^{\frac{2^n-1}{5}})$, where $n = 2m$, $m \equiv 2 \pmod{4}$, $a \in \mathbb{F}_{2^m}$ and $b \in \mathbb{F}_{16}$. When $a \in \mathbb{F}_{2^m}$ and $(b+1)(b^4+b+1) = 0$, with the help of Kloosterman sums and the factorization of $x^5 + x + a^{-1}$, we present a characterization of hyper-bentness of $f_{a,b}$. Further, we use generalized Ramanujan-Nagell equations to characterize hyper-bent functions of $f_{a,b}$ in the case $a \in \mathbb{F}_{2^{\frac{m}{2}}}$.

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1 Introduction

Bent functions are maximally nonlinear Boolean functions with even numbers of variables whose Hamming distance to the set of all affine functions equals $2^{n-1} \pm 2^{\frac{n}{2}-1}$. These functions introduced by Rothaus [35] as interesting combinatorial objects have been extensively studied for their applications not only in cryptography, but also in coding theory [5,31] and combinatorial design. Some basic knowledge and recent results on bent functions can be found in [4,13,31]. A bent function can be considered as a Boolean function defined over \mathbb{F}_2^n , $\mathbb{F}_{2^m} \times \mathbb{F}_{2^m}$ ($n = 2m$) or \mathbb{F}_{2^n} . Thanks to the different structures of the vector space \mathbb{F}_2^n and the Galois field \mathbb{F}_{2^n} , bent functions can be well studied. Although some algebraic properties of bent functions are well known, the general structure of bent functions on \mathbb{F}_{2^n} is not clear yet. As a result, much research on bent functions on \mathbb{F}_{2^n} can be found in [2,8,9,11,12,14,15,24,25,29,30,31,32,33,38]. Youssef and Gong [37] introduced a class of bent functions called hyper-bent functions, which achieve the maximal minimum distance to all the coordinate functions of all bijective monomials (i.e., functions of the form $\text{Tr}_1^n(ax^i) + \epsilon$, $\gcd(i, 2^n - 1) = 1$). However, the definition of hyper-bent functions was given by Gong and Golomb [16] by a property of the extend Hadamard transform of Boolean functions. Hyper-bent functions as special bent functions with strong properties are hard to characterize and many related problems are open. Much research give the precise characterization of hyper-bent functions in certain forms.

The complete classification of bent and hyper-bent functions is not yet achieved. The monomial bent functions in the form $\text{Tr}_1^n(ax^s)$ are considered in [2,24]. Leander [24] described the necessary conditions for s such that $\text{Tr}_1^n(ax^s)$ is a bent function. In particular, when $s = r(2^m - 1)$ and $(r, 2^m + 1) = 1$, the monomial functions $\text{Tr}_1^n(ax^s)$ (i.e., the Dillon functions) were extensively studied in [8,11,24]. A class of quadratic functions over \mathbb{F}_{2^n} in polynomial form $\sum_{i=1}^{\frac{n}{2}-1} a_i \text{Tr}_1^n(x^{1+2^i}) + a_{\frac{n}{2}} \text{Tr}_1^{\frac{n}{2}}(x^{\frac{n}{2}+1})$ ($a_i \in \mathbb{F}_2$) was described and studied in [10,18,19,20,26,38]. Dobbertin et al. [14] constructed a class of binomial bent functions of the form $\text{Tr}_1^n(a_1x^{s_1} + a_2x^{s_2})$, $(a_1, a_2) \in (\mathbb{F}_{2^n}^*)^2$ with Niho power functions. Garlet and Mesnager [7] studied the duals of the Niho bent functions in [14]. In [29,30,33], Mesnager considered the binomial functions of the form $\text{Tr}_1^n(ax^{r(2^m-1)}) + \text{Tr}_1^2(bx^{\frac{2^n-1}{3}})$, where $a \in \mathbb{F}_{2^n}^*$ and $b \in \mathbb{F}_4^*$. Then he gave the link between the bentness property of such functions and Kloosterman sums. Leander and Kholosha [25] generalized one of the constructions proven by Dobbertin et al. [14] and presented a new primary construction of bent functions consisting of a linear combination of 2^r Niho exponents. Carlet et al. [6] computed the dual of the Niho bent function with 2^r exponents found

by Leander and Kholosha [25] and showed that this new bent function is not of the Niho type. Charpin and Gong [8] presented a characterization of bentness of Boolean functions over \mathbb{F}_{2^n} of the form $\sum_{r \in R} \text{Tr}_1^n(a_r x^{r(2^m-1)})$, where

R is a subset of the set of representatives of the cyclotomic cosets modulo $2^m + 1$ of maximal size n . These functions include the well-known monomial functions with the Dillon exponent as a special case. Then they described the bentness of these functions with the Dickson polynomials. Mesnager et al. [31, 32] generalized the results of Charpin and Gong [8] and considered the bentness of Boolean functions over \mathbb{F}_{2^n} of the form $\sum_{r \in R} \text{Tr}_1^n(a_r x^{r(2^m-1)}) + \text{Tr}_1^2(bx^{\frac{2^n-1}{3}})$,

where $n = 2m$, $a_r \in \mathbb{F}_{2^m}$ and $b \in \mathbb{F}_4$. Further, they presented the link between the bentness of such functions and some exponential sums (involving Dickson polynomials).

In this paper, we consider a class of Boolean functions \mathcal{H}_n . A Boolean function $f_{a,b}$ in \mathcal{H}_n is defined over \mathbb{F}_{2^n} by the form: $f_{a,b} := \text{Tr}_1^n(ax^{(2^m-1)}) + \text{Tr}_1^4(bx^{\frac{2^n-1}{5}})$, where $n = 2m$, $m \equiv 2 \pmod{4}$, $a \in \mathbb{F}_{2^m}$ and $b \in \mathbb{F}_{16}$. When $b = 0$, Charpin and Gong [8] described the bentness and hyper-bentness of these functions with some character sums involving Dickson polynomial. Generally, it is elusive to give a characterization of bentness and hyper-bentness of Boolean functions in \mathcal{H}_n . This paper discusses the hyper-bentness of Boolean function in \mathcal{H}_n for two cases. In the first case $b = 1$ or $b^4 + b + 1 = 0$, we present the hyper-bentness of $f_{a,b}$ by the factorization of $x^5 + x + a^{-1}$ and Kloosterman sums. For the second case $a \in \mathbb{F}_{2^{\frac{m}{2}}}$, we give all the hyper-bent functions with the help of generalized Ramanujan-Nagell equations.

The rest of paper is organized as follows. In Section 2, we give some notations and recall some basic knowledge for the paper. In Section 3, we study the hyper-bentness of Boolean functions in \mathcal{H}_n for two cases (1) $b = 1$ or $b^4 + b + 1 = 0$; (2) $a \in \mathbb{F}_{2^{\frac{m}{2}}}$. Finally, Section 4 makes a conclusion.

2 Preliminaries

2.1 Boolean functions

Let n be a positive integer. \mathbb{F}_2^n is a n -dimensional vector space defined over finite field \mathbb{F}_2 . Take two vectors $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in \mathbb{F}_2^n . Their dot product is defined by

$$\langle x, y \rangle := \sum_{i=1}^n x_i y_i.$$

\mathbb{F}_{2^n} is a finite field with 2^n elements and $\mathbb{F}_{2^n}^*$ is the multiplicative group of \mathbb{F}_{2^n} . Let \mathbb{F}_{2^k} be a subfield of \mathbb{F}_{2^n} . The trace function from \mathbb{F}_{2^n} to \mathbb{F}_{2^k} , denoted by Tr_k^n , is a map defined as

$$\text{Tr}_k^n(x) := x + x^{2^k} + x^{2^{2k}} + \dots + x^{2^{n-k}}.$$

When $k = 1$, Tr_1^n is called the absolute trace. The trace function Tr_k^n satisfies the following properties.

$$\begin{aligned}\text{Tr}_k^n(ax + by) &= a\text{Tr}_k^n(x) + b\text{Tr}_k^n(y), \quad a, b \in \mathbb{F}_{2^k}, x, y \in \mathbb{F}_{2^n}. \\ \text{Tr}_k^n(x^{2^k}) &= \text{Tr}_k^n(x), \quad x \in \mathbb{F}_{2^n}.\end{aligned}$$

When $\mathbb{F}_{2^k} \subseteq \mathbb{F}_{2^r} \subseteq \mathbb{F}_{2^n}$, the trace function Tr_k^n satisfies the following transitivity property.

$$\text{Tr}_k^n(x) = \text{Tr}_k^r(\text{Tr}_r^n(x)), \quad x \in \mathbb{F}_{2^n}.$$

A Boolean function over \mathbb{F}_2^n or \mathbb{F}_{2^n} is an \mathbb{F}_2 -valued function. The absolute trace function is a useful tool in constructing Boolean functions over \mathbb{F}_{2^n} . From the absolute trace function, a dot product over \mathbb{F}_{2^n} is defined by

$$\langle x, y \rangle := \text{Tr}_1^n(xy), \quad x, y \in \mathbb{F}_{2^n}.$$

A Boolean function over \mathbb{F}_{2^n} is often represented by the algebraic normal form (ANF):

$$f(x_1, \dots, x_n) = \sum_{I \subseteq \{1, \dots, n\}} a_I \left(\prod_{i \in I} x_i \right), \quad a_I \in \mathbb{F}_2.$$

When $I = \emptyset$, let $\prod_{i \in I} x_i = 1$. The terms $\prod_{i \in I} x_i$ are called monomials. The algebraic degree of a Boolean function f is the globe degree of its ANF, that is, $\deg(f) := \max\{\#(I) | a_I \neq 0\}$, where $\#(I)$ is the order of I and $\#(\emptyset) = 0$.

Another representation of a Boolean function is of the form

$$f(x) = \sum_{j=0}^{2^n-1} a_j x^j.$$

In order to make f a Boolean function, we should require $a_0, a_{2^n-1} \in \mathbb{F}_2$ and $a_{2j} = a_j^2$, where $2j$ is taken modulo $2^n - 1$. This makes that f can be represented by a trace expansion of the form

$$f(x) = \sum_{j \in \Gamma_n} \text{Tr}_1^{o(j)}(a_j x^j) + \epsilon(1 + x^{2^n-1})$$

called its polynomial form, where

- Γ_n is the set of integers obtained by choosing one element in each cyclotomic class of 2 modulo $2^n - 1$ (j is often chosen as the smallest element in its cyclotomic class, called the coset leader of the class);
- $o(j)$ is the size of the cyclotomic coset of 2 modulo $2^n - 1$ containing j ;
- $a_j \in \mathbb{F}_{2^{o(j)}}$;
- $\epsilon = \text{wt}(f) \pmod{2}$, where $\text{wt}(f) := \#\{x \in \mathbb{F}_{2^n} | f(x) = 1\}$.

Let $\text{wt}_2(j)$ be the number of 1's in its binary expansion. Then

$$\deg(f) = \begin{cases} n, & \epsilon = 1 \\ \max\{\text{wt}_2(j) | a_j \neq 0\}, & \epsilon = 0. \end{cases}$$

2.2 Bent and hyper-bent functions

The "sign" function of a Boolean function f is defined by

$$\chi(f) := (-1)^f.$$

When f is a Boolean function over \mathbb{F}_2^n , the Walsh Hadamard transform of f is the discrete Fourier transform of $\chi(f)$, whose value at $w \in \mathbb{F}_2^n$ is defined by

$$\widehat{\chi}_f(w) := \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) + \langle w, x \rangle}.$$

When f is a Boolean function over \mathbb{F}_{2^n} , the Walsh Hadamard transform of f is defined by

$$\widehat{\chi}_f(w) := \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x) + \text{Tr}_1^n(wx)},$$

where $w \in \mathbb{F}_{2^n}$. Then we can define the bent functions.

Definition 1 A Boolean function $f : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_2$ is called a bent function, if $\widehat{\chi}_f(w) = \pm 2^{\frac{n}{2}}$ ($\forall w \in \mathbb{F}_{2^n}$).

If f is a bent function, n must be even. Further, $\deg(f) \leq \frac{n}{2}$ [4]. Hyper-bent functions are an important subclass of bent functions. The definition of hyper-bent functions is given below.

Definition 2 A bent function $f : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_2$ is called a hyper-bent function, if, for any i satisfying $(i, 2^n - 1) = 1$, $f(x^i)$ is also a bent function.

[5] and [37] proved that if f is a hyper-bent function, then $\deg(f) = \frac{n}{2}$. For a bent function f , $\text{wt}(f)$ is even. Then $\epsilon = 0$, that is,

$$f(x) = \sum_{j \in \Gamma_n} \text{Tr}_1^{o(j)}(a_j x^j).$$

If a Boolean function f is defined on $\mathbb{F}_{2^{\frac{n}{2}}} \times \mathbb{F}_{2^{\frac{n}{2}}}$, then we have a class of bent functions [11, 27].

Definition 3 The Maiorana-McFarland class \mathcal{M} is the set of all the Boolean functions f defined on $\mathbb{F}_{2^{\frac{n}{2}}} \times \mathbb{F}_{2^{\frac{n}{2}}}$ of the form $f(x, y) = \langle x, \pi(y) \rangle + g(y)$, where $x, y \in \mathbb{F}_{2^{\frac{n}{2}}}$, π is a permutation of $\mathbb{F}_{2^{\frac{n}{2}}}$ and $g(x)$ is a Boolean function over $\mathbb{F}_{2^{\frac{n}{2}}}$.

For Boolean functions over $\mathbb{F}_{2^{\frac{n}{2}}} \times \mathbb{F}_{2^{\frac{n}{2}}}$, we have a class of hyper-bent functions \mathcal{PS}_{ap} [5].

Definition 4 Let $n = 2m$, the \mathcal{PS}_{ap} class is the set of all the Boolean functions of the form $f(x, y) = g(\frac{x}{y})$, where $x, y \in \mathbb{F}_{2^m}$ and g is a balanced Boolean functions (i.e., $\text{wt}(f) = 2^{m-1}$) and $g(0) = 0$. When $y = 0$, let $\frac{x}{y} = xy^{2^n-2} = 0$.

Each Boolean function f in \mathcal{PS}_{ap} satisfies $f(\beta z) = f(z)$ and $f(0) = 0$, where $\beta \in \mathbb{F}_m^*$ and $z \in \mathbb{F}_m \times \mathbb{F}_m$. Youssef and Gong [37] studied these functions over \mathbb{F}_{2^n} and gave the following property.

Proposition 1 Let $n = 2m$, α be a primitive element in \mathbb{F}_{2^n} and f be a Boolean function over \mathbb{F}_{2^n} such that $f(\alpha^{2^m+1}x) = f(x) (\forall x \in \mathbb{F}_{2^n})$ and $f(0) = 0$, then f is a hyper-bent function if and only if the weight of $(f(1), f(\alpha), f(\alpha^2), \dots, f(\alpha^{2^m}))$ is 2^{m-1} .

Further, [5] proved the following result.

Proposition 2 Let f be a Boolean function defined in Proposition 1. If $f(1) = 0$, then f is in \mathcal{PS}_{ap} . If $f(1) = 1$, then there exists a Boolean function g in \mathcal{PS}_{ap} and $\delta \in \mathbb{F}_{2^n}^*$ satisfying $f(x) = g(\delta x)$.

Let $\mathcal{PS}_{ap}^\#$ be the set of hyper-bent functions in the form of $g(\delta x)$, where $g(x) \in \mathcal{PS}_{ap}$, $\delta \in \mathbb{F}_{2^n}^*$ and $g(\delta) = 1$. Charpin and Gong expressed Proposition 2 in a different version below.

Proposition 3 Let $n = 2m$, α be a primitive element of \mathbb{F}_{2^n} and f be a Boolean function over \mathbb{F}_{2^n} satisfying $f(\alpha^{2^m+1}x) = f(x) (\forall x \in \mathbb{F}_{2^n})$ and $f(0) = 0$. Let ξ be a primitive $2^m + 1$ -th root in $\mathbb{F}_{2^n}^*$. Then f is a hyper-bent function if and only if the cardinality of the set $\{i | f(\xi^i) = 1, 0 \leq i \leq 2^m\}$ is 2^{m-1} .

In fact, Dillon [11] introduced a bigger class of bent functions the Partial Spreads class \mathcal{PS}^- than \mathcal{PS}_{ap} and $\mathcal{PS}_{ap}^\#$.

Theorem 1 Let $E_i (i = 1, 2, \dots, N)$ be N subspaces in \mathbb{F}_{2^n} of dimension m such that $E_i \cap E_j = \{0\}$ for all $i, j \in \{1, \dots, N\}$ with $i \neq j$. Let f be a Boolean function over \mathbb{F}_{2^n} . If the support of f is given by $\text{supp}(f) = \bigcup_{i=1}^N E_i^*$, where $E_i^* = E_i \setminus \{0\}$, then f is a bent function if and only if $N = 2^{m-1}$.

The set of all the functions in Theorem 1 is defined by \mathcal{PS}^- .

2.3 Dickson polynomials

Now we recall the knowledge of Dickson polynomials over \mathbb{F}_2 [34]. For $r > 0$, Dickson polynomials are given by

$$D_r(x) = \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} \frac{r}{r-i} \binom{r-i}{i} x^{r-2i}, r = 2, 3, \dots$$

Further, Dickson polynomials can be also defined by the following recurrence relation.

$$D_{i+2}(x) = xD_{i+1} + D_i(x)$$

with initial values

$$D_0(x) = 0, D_1(x) = x.$$

Some properties of Dickson polynomials are given below.

- $\deg(D_r(x)) = r$.
- $D_{rp}(x) = D_r(D_p(x))$.

$$- D_r(x + x^{-1}) = x^r + x^{-r}.$$

The first few Dickson polynomials with odd r are

$$\begin{aligned} D_1(x) &= x, \\ D_3(x) &= x + x^3, \\ D_5(x) &= x + x^3 + x^5, \\ D_7(x) &= x + x^5 + x^7, \\ D_9(x) &= x + x^5 + x^7 + x^9, \\ D_{11}(x) &= x + x^3 + x^5 + x^9 + x^{11}. \end{aligned}$$

The following proposition gives some properties on Dickson polynomials [12, 34].

Proposition 4 *Let m and k be two positive integers. Let $x_0, y_0 \in \mathbb{F}_{2^m}$ and $y_0 = D_k(x_0)$, then*

$$\#\{x \in \mathbb{F}_{2^m} \mid D_k(x) = y_0\} = \begin{cases} d_1 & \text{If } x^2 + x_0x + 1 \text{ is irreducible over } \mathbb{F}_{2^m} \text{ and } y_0 \neq 0 \\ d_2 & \text{If } x^2 + x_0x + 1 \text{ is reducible over } \mathbb{F}_{2^m} \text{ and } y_0 \neq 0 \\ \frac{d_1 + d_2}{2} & \text{If } y_0 = 0 \end{cases}$$

where $d_1 = (k, 2^m - 1)$ and $d_2 = (k, 2^m + 1)$.

The reducibility of the polynomial $x^2 + x_0x + 1$ can be determined by the following proposition.

Proposition 5 *Let m be a positive integer and $x_0 \in \mathbb{F}_{2^m}$. Then $x^2 + x_0x + 1$ is reducible over \mathbb{F}_{2^m} if and only if $\text{Tr}_1^m(\frac{1}{x_0}) = 0$.*

2.4 Kloosterman sums and Weil sums

The Kloosterman sums on \mathbb{F}_{2^n} are:

$$K_m(a) := \sum_{x \in \mathbb{F}_{2^m}} \chi(\text{Tr}_1^m(ax + \frac{1}{x})), \quad a \in \mathbb{F}_{2^m}.$$

Some properties of Kloosterman sums are given by the following proposition [17, 21].

Proposition 6 *Let $a \in \mathbb{F}_{2^m}$. Then $K_m(a) \in [1 - 2^{(m+2)/2}, 1 + 2^{(m+2)/2}]$ and $4 \mid K_m(a)$.*

Weil sums of degree 5 on \mathbb{F}_{2^m} are:

$$Q_m(a) := \sum_{x \in \mathbb{F}_{2^m}} \chi(\text{Tr}_1^m(a(x^5 + x^3 + x))), \quad a \in \mathbb{F}_{2^m}.$$

To determine the value of $Q_m(a)$, we introduce an affine Artin-Schreier model of the form [28, 36]:

$$C : y^2 + y = a(x^5 + x^3 + x), a \in \mathbb{F}_{2^m}^*.$$

This curve is a smooth supersingular curve of genus 2 over \mathbb{F}_{2^m} and it has only one point at infinity. Let $J(C)$ be the Jacobian of C over \mathbb{F}_{2^m} . Let $f_{J(C)}(x)$ be the Weil polynomial for $J(C)$ of the form:

$$f_{J(C)}(x) = x^4 + rx^3 + sx^2 + 2^m rx + 2^n,$$

where (r, s) is determined by the irreducible factors of the polynomial $P(x) = x^5 + x + a^{-1}$. We write that $P(x) = (n_1)^{r_1}(n_2)^{r_2} \cdots (n_t)^{r_t}$ to indicate that r_i of the irreducible factors of $P(x)$ have degree n_i . Some results on the Weil polynomial of C are given in the following proposition [3].

Proposition 7 *Let $P(x)$, r and s be defined above, then*

- (1) *If m is even, (r, s) is determined by Table 1.*

Table 1 $P(x)$ and (r, s) for even m

$P(x)$	(r, s)
(5)	$(\pm 2^{m/2}, 2^{m/2})$
$(1)^2(3)$	$(\pm 2 \cdot 2^{m/2}, 3 \cdot 2^m)$
$(1)(2)^2$	$(0, 2 \cdot 2^m)$
$(1)^5$	$(\pm 4 \cdot 2^{m/2}, 6 \cdot 2^m)$

- (2) *If m is odd, (r, s) is determined by Table 2.*

Table 2 $P(x)$ and (r, s) for odd m

$P(x)$	(r, s)
$(1)(4)$	$(0, 0)$
$(2)(3)$	$(0, 2^m)$
$(1)^3(2)$	$(0, -2 \cdot 2^m)$

The number of rational points on C (including the infinite point) can be determined by the Weil polynomial $f_{J(C)}(x)$. We write $N(\mathbb{F}_{2^m}) = \#(C(\mathbb{F}_{2^m}))$. Then [28]

$$N(\mathbb{F}_{2^m}) = 2^m + 1 + r, N(\mathbb{F}_{2^{2m}}) = 2^{2m} + 1 + 2s - r^2. \quad (1)$$

Further, we have the following result.

Proposition 8 *Let $Q_m(a)$ and r be defined above, then*

$$Q_m(a) = r.$$

Proof To prove $Q_m(a) = r$, we just prove that

$$Q_m(a) = N(F_{2^m}) - (2^m + 1) = \#(C(\mathbb{F}_{2^m})) - (2^m + 1).$$

$C(\mathbb{F}_{2^m})$ has only an infinite point. A point (x, y) is in $C(\mathbb{F}_{2^m})$ if and only if $x \in \mathbb{F}_{2^m}$ and $\text{Tr}_1^m(a(x^5 + x^3 + x)) = 0$. Then

$$\begin{aligned} \#(C(\mathbb{F}_{2^m})) &= 1 + 2 \cdot \#\{x \in \mathbb{F}_{2^m} \mid \text{Tr}_1^m(a(x^5 + x^3 + x)) = 0\} \\ &= 2 \cdot \#\{x \in \mathbb{F}_{2^m} \mid \text{Tr}_1^m(a(x^5 + x^3 + x)) = 0\} - 2^m + (2^m + 1) \\ &= \sum_{x \in \mathbb{F}_{2^m}} \chi(\text{Tr}_1^m(a(x^5 + x^3 + x))) + (2^m + 1). \end{aligned}$$

From the definition of $Q_m(a)$,

$$\#(C(\mathbb{F}_{2^m})) = 2^m + 1 + Q_m(a).$$

Hence, $Q_m(a) = r$.

From Proposition 7 and Proposition 8, we can easily obtain two following corollaries.

Corollary 1 *Let m be even and $a \in \mathbb{F}_{2^m}$, then $Q_m(a) \in \{0, \pm 2^{m/2}, \pm 2 \cdot 2^{m/2}, \pm 4 \cdot 2^{m/2}\}$.*

Corollary 2 *Let m be even and $a \in \mathbb{F}_{2^m}^*$, then*

- (1) $Q_m(a) = 0$ if and only if $P(x) = x^5 + x + a^{-1} = (1)(2)^2$.
- (2) $Q_m(a) = \pm 2^{m/2}$ if and only if $P(x) = x^5 + x + a^{-1}$ is irreducible over \mathbb{F}_{2^m} .

Remark 1 When $P(x) = x^5 + x + a^{-1}$ is irreducible over \mathbb{F}_{2^m} , the sign of $Q_m(a)$ is related to the parity of the quadratic form $\mathbf{q}(x) = \text{Tr}_1^m(x(ax^4 + ax^2 + a^2x))$. $\mathbf{q}(x)$ is the quadratic form associated to the symplectic form:

$$\langle x, y \rangle_{\mathbf{q}} := \text{Tr}_1^m(x(ay^4 + ay^2 + a^2y) + y(ax^4 + ax^2 + a^2x)),$$

which is non-degenerate. Then there exists a normal symplectic basis $e_1, e_{m_1+1}, \dots, e_{m_1}, e_{2m_1}$ ($2m_1 = m$). If $i \not\equiv j \pmod{m_1}$, $\langle e_i, e_j \rangle_{\mathbf{q}} = 0$. For any i ($1 \leq i \leq m_1$), $\langle e_i, e_{m_1+i} \rangle_{\mathbf{q}} = 1$. If $\#\{i \mid \mathbf{q}(e_i) = \mathbf{q}(e_{m_1+i}) = 1, 1 \leq i \leq m_1\}$ is even, then the quadratic form $\mathbf{q}(x)$ is even and $Q_m(a) = 2^{m_1}$. If $\#\{i \mid \mathbf{q}(e_i) = \mathbf{q}(e_{m_1+i}) = 1, 1 \leq i \leq m_1\}$ is odd, then the quadratic form $\mathbf{q}(x)$ is odd and $Q_m(a) = -2^{m_1}$. In fact, the parity of $\mathbf{q}(x)$ can be determined by the point multiplication of a random element in $J(C)(\mathbb{F}_{2^m})$. Consequently, the sign of $Q_m(a)$ is determined [3].

3 A class of hyper-bent functions in binomial forms

3.1 Boolean functions in \mathcal{H}_n

Through the paper, we assume that $n = 2m$ and $m \equiv 2 \pmod{4}$. Let \mathcal{H}_n be the set of Boolean functions on \mathbb{F}_{2^n} of the form:

$$f_{a,b}(x) := \text{Tr}_1^n(ax^{2^m-1}) + \text{Tr}_1^4(bx^{\frac{2^n-1}{5}}), \quad (2)$$

where $a \in \mathbb{F}_{2^m}$ and $b \in \mathbb{F}_{16}$.

Note that the cyclotomic coset of 2 module $2^n - 1$ containing $\frac{2^n-1}{5}$ is $\{\frac{2^n-1}{5}, 2 \cdot \frac{2^n-1}{5}, 2^2 \cdot \frac{2^n-1}{5}, 2^3 \cdot \frac{2^n-1}{5}\}$. Then its size is 4, that is, $o(\frac{2^n-1}{5}) = 4$. Hence, the Boolean function $f_{a,b}$ is not in the class considered by Charpin and Gong [6]. Further, it does not lie in the class of Boolean functions studied by Mesnager [31, 32].

Since $m \equiv 2 \pmod{4}$ and $2^m + 1 \equiv 0 \pmod{5}$, any Boolean function $f_{a,b}$ in \mathcal{H}_n satisfies

$$f_{a,b}(\alpha^{2^m+1}x) = f_{a,b}(x), \forall x \in \mathbb{F}_{2^n},$$

where α is a primitive element in \mathbb{F}_{2^n} . Note that $f_{a,b}(0) = 0$. Then the hyper-bentness $f_{a,b}$ can be characterized by the following proposition.

Proposition 9 *Let $f_{a,b} \in \mathcal{H}_n$. Set the following sum:*

$$\Lambda(a,b) := \sum_{u \in U} \chi(f_{a,b}(u)) \quad (3)$$

where U is the group of all $2^m + 1$ -th roots of unity in \mathbb{F}_{2^n} , that is, $U = \{x \in \mathbb{F}_{2^n} | x^{2^m+1} = 1\}$. Then $f_{a,b}$ is a hyper-bent function if and only if $\Lambda(a,b) = 1$. Further, a hyper-bent function $f_{a,b}$ lies in \mathcal{PS}_{ap} if and only if $\text{Tr}_1^4(b) = 0$.

Proof From Proposition 3, $f_{a,b}$ is a hyper-bent function if and only if its restriction to U has Hamming weight 2^{m-1} . Note that

$$\begin{aligned} \Lambda(a,b) &= \sum_{x \in U} \chi(f_{a,b}(x)) \\ &= \#\{u \in U | f_{a,b}(u) = 0\} - \#\{u | f_{a,b}(u) = 1\} \\ &= \#U - 2\#\{u | f_{a,b}(u) = 1\} \\ &= 2^m + 1 - 2\#\{u | f_{a,b}(u) = 1\}. \end{aligned}$$

Then the restriction of $f_{a,b}$ to U has Hamming weight 2^{m-1} if and only if $\Lambda(f_{a,b}) = 1$. Hence, $f_{a,b}$ is a hyper-bent if and only if $\Lambda(f_{a,b}) = 1$. As for the second part of this proposition, we get that

$$\begin{aligned} f_{a,b}(1) &= \text{Tr}_1^n(a) + \text{Tr}_1^4(b) \\ &= \text{Tr}_1^m(a + a^{2^m}) + \text{Tr}_1^4(b) \\ &= \text{Tr}_1^4(b). \end{aligned}$$

Then $f_{a,b}(1) = 0$ if and only if $\text{Tr}_1^4(b) = 0$. Hence, from Proposition 3, $f_{a,b}$ lies in \mathcal{PS}_{ap} if and only if $\text{Tr}_1^4(b) = 0$.

3.2 Character sums on Boolean functions in \mathcal{H}_n

Dillon [11] presented the characterization of the hyper-bentness of $f_{a,0}$. In this paper, we consider the hyper-bentness of $f_{a,b}$ ($b \neq 0$) in \mathcal{H}_n . We first give some notations and present some properties of character sums.

Let α be a primitive element in \mathcal{H}_n . Then $\beta = \alpha^{\frac{2^n-1}{5}}$ is a primitive 5-th root of unity in U , where U is the cyclic group generated by $\xi = \alpha^{2^m-1}$. Let V be the cyclic group generated by $\alpha^{5(2^m-1)}$. Then we have

$$U = \cup_{i=0}^4 \xi^i V, \quad \mathbb{F}_{2^n}^* = \mathbb{F}_{2^m}^* \times U.$$

We introduce the character sums:

$$S_i = \sum_{v \in V} \chi(\text{Tr}_1^n(a(\xi^i v)^{2^m-1})).$$

Then, we have

$$S_0 + S_1 + S_2 + S_3 + S_4 = \sum_{u \in U} \chi(\text{Tr}_1^n(au^{2^m-1})) = \Lambda(a, 0). \quad (4)$$

Obviously, for any integer i , $S_i = S_{i \pmod{5}}$. Further, we have the following lemma on S_i .

Lemma 1 $S_1 = S_4$, $S_2 = S_3$.

Proof Note that $\text{Tr}_1^n(x^{2^m}) = \text{Tr}_1^n(x)$, then

$$S_i = \sum_{v \in V} \chi(\text{Tr}_1^n(a(\xi^i v)^{(2^m-1)})) = \sum_{v \in V} \chi(\text{Tr}_1^n(a^{2^m}(\xi^{i2^m} v^{2^m})^{(2^m-1)})).$$

From $a \in \mathbb{F}_{2^m}$, $a^{2^m} = a$. Since $m \equiv 2 \pmod{4}$ and $2^m \equiv -1 \pmod{5}$, hence $i2^m \equiv -i \pmod{5}$ and $\xi^{i2^m} v^{2^m} = \xi^{-i}(\xi^{i(2^m+1)} v^{2^m})$, where $\xi^{i(2^m+1)} \in V$. The map $v \mapsto \xi^{i(2^m+1)} v^{2^m}$ is a permutation of V . Consequently,

$$S_i = \sum_{v \in V} \chi(\text{Tr}_1^n(a(\xi^{-i} v)^{(2^m-1)})) = S_{-i}.$$

We just take $i = 1, 2$. Then this lemma follows.

From (4) and Lemma 1, we can get the following corollary.

Corollary 3 $S_0 + 2(S_1 + S_2) = \Lambda(a, 0)$.

Further, $\Lambda(a, b)$ is a linear combination of S_0 , S_1 and S_2 .

Proposition 10 $\Lambda(a, b) = \chi(\text{Tr}_1^4(b))S_0 + (\chi(\text{Tr}_1^4(b\beta^2)) + \chi(\text{Tr}_1^4(b\beta^3)))S_1 + (\chi(\text{Tr}_1^4(b\beta)) + \chi(\text{Tr}_1^4(b\beta^4)))S_2$.

Proof From (3), we have

$$\begin{aligned}
\Lambda(a, b) &= \sum_{u \in U} \chi(f_{a,0}(u) + \text{Tr}_1^4(bu^{\frac{2^n-1}{5}})) \\
&= \sum_{u \in U} \chi(\text{Tr}_1^4(bu^{\frac{2^n-1}{5}})) \chi(f_{a,0}(u)) \\
&= \sum_{i=0}^4 \sum_{v \in V} \chi(\text{Tr}_1^4(b(\xi^i v)^{\frac{2^n-1}{5}})) \chi(f_{a,0}(\xi^i v)) \quad (\text{From (4)}) \\
&= \sum_{i=0}^4 \sum_{v \in V} \chi(\text{Tr}_1^4(b(\alpha^{i(2^m-1)})^{\frac{2^n-1}{5}})) \chi(f_{a,0}(\xi^i v)) \quad (\xi = \alpha^{2^m-1}) \\
&= \sum_{i=0}^4 \sum_{v \in V} \chi(\text{Tr}_1^4(b\beta^{i(2^m-1)})) \chi(f_{a,0}(\xi^i v)) \quad (\beta = \alpha^{\frac{2^n-1}{5}}).
\end{aligned}$$

Since $2^m + 1 \equiv 0 \pmod{5}$, $2^m - 1 \equiv 3 \pmod{5}$ and

$$\Lambda(a, b) = \sum_{i=0}^4 \sum_{v \in V} \chi(\text{Tr}_1^4(b\beta^{3i})) \chi(f_{a,0}(\xi^i v)) = \sum_{i=0}^4 \chi(\text{Tr}_1^4(b\beta^{3i})) \sum_{v \in V} \chi(f_{a,0}(\xi^i v)).$$

From the definition of S_i ,

$$\Lambda(a, b) = \sum_{i=0}^4 \chi(\text{Tr}_1^4(b\beta^{3i})) S_i.$$

From Lemma 1, this proposition follows.

Further, if $a \in \mathbb{F}_{2^{m_1}}$, where $m_1 = m/2$, we have the following proposition.

Proposition 11 *If $a \in \mathbb{F}_{2^{m_1}}$, where $m_1 = m/2$, then*

$$S_1 = S_2, S_0 + 4S_1 = \Lambda(a, 0).$$

Proof From $a \in \mathbb{F}_{2^{m_1}}$, $\text{Tr}_1^n(ax^{2^{m_1}-1}) = \text{Tr}_1^n(ax^{2^{m_1}(2^m-1)})$. Then

$$S_i = \sum_{v \in V} \chi(\text{Tr}_1^n(a(\xi^i v)^{(2^m-1)})) = \sum_{v \in V} \chi(\text{Tr}_1^n(a(\xi^{2^{m_1}i} v^{2^{m_1}})^{(2^m-1)})).$$

We take $i = 1$. Then

$$S_1 = \sum_{v \in V} \chi(\text{Tr}_1^n(a(\xi^{2^{m_1}} v^{2^{m_1}})^{(2^m-1)})).$$

Since $2^m + 1 \equiv 0 \pmod{5}$, $(2^{m_1})^2 \equiv -1 \pmod{5}$ and $2^{m_1} \equiv \pm 2 \pmod{5}$.

When $2^{m_1} \equiv 2 \pmod{5}$,

$$S_1 = \sum_{v \in V} \chi(\text{Tr}_1^n(a(\xi^2 \xi^{2^{m_1}-2} v^{2^{m_1}})^{(2^m-1)})).$$

The map $v \mapsto \xi^{2^{m_1}-2}v^{2^{m_1}}$ is a permutation of V . Consequently,

$$S_1 = \sum_{v \in V} \chi(\text{Tr}_1^n(a(\xi^2 v)^{(2^m-1)})) = S_2.$$

When $2^{m_1} \equiv -2 \pmod{5}$, we can similarly have $S_1 = S_3$.

Above all, $S_1 = S_2$. From Corollary 3, $S_0 + 4S_1 = \Lambda(a, 0)$.

For $\Lambda(a, b)$, the following proposition gives some properties.

Proposition 12 $\Lambda(a, b)$ satisfies

- (1) $\Lambda(a, b^4) = \Lambda(a, b)$.
- (2) If b is a primitive element of $\mathbb{F}_{2^{16}}$ and $\text{Tr}_1^4(b) = 0$, $\Lambda(a, b^2) = \Lambda(a, b) = S_0$.

Proof (1) For any $b \in \mathbb{F}_{16}$, $\text{Tr}_1^4(b^4) = \text{Tr}_1^4(b)$. Then

$$\begin{aligned} \text{Tr}_1^4(b(\beta^2 + \beta^3)) &= \text{Tr}_1^4(b^4(\beta^8 + \beta^{12})) = \text{Tr}_1^4(b^4(\beta^2 + \beta^3)) \\ \text{Tr}_1^4(b(\beta + \beta^4)) &= \text{Tr}_1^4(b^4(\beta^4 + \beta^{16})) = \text{Tr}_1^4(b^4(\beta + \beta^4)). \end{aligned}$$

From Proposition 10, $\Lambda(a, b^4) = \Lambda(a, b)$.

(2) If b is a primitive element in \mathbb{F}_{16} such that $\text{Tr}_1^4(b) = 0$, then $b^4 + b + 1 = 0$. Further,

$$\begin{aligned} \text{Tr}_1^4(b(\beta^2 + \beta^3)) &= \text{Tr}_1^2(b^4(\beta^2 + \beta^3) + b(\beta^2 + \beta^3)) \\ &= \text{Tr}_1^2((b + b^4)(\beta^2 + \beta^3)) \\ &= \text{Tr}_1^2(\beta^2 + \beta^3). \end{aligned}$$

The minimal polynomial of β in \mathbb{F}_2 is $\beta^4 + \beta^3 + \beta^2 + \beta + 1 = 0$. Then

$$\text{Tr}_1^2(\beta^2 + \beta^3) = \beta^2 + \beta^3 + \beta^4 + \beta^6 = 1.$$

Consequently, $\text{Tr}_1^4(b(\beta^2 + \beta^3)) = 1$. Similarly, $\text{Tr}_1^4(b(\beta + \beta^4)) = 1$. Therefore,

$$\chi(\text{Tr}_1^4(b\beta^2)) + \chi(\text{Tr}_1^4(b\beta^3)) = 0, \quad \chi(\text{Tr}_1^4(b\beta)) + \chi(\text{Tr}_1^4(b\beta^4)) = 0.$$

From Proposition 10, $\Lambda(a, b) = S_0$.

Since b is a primitive element such that $\text{Tr}_1^4(b) = 0$, b^2 is also a primitive element such that $\text{Tr}_1^4(b^2) = 0$. Consequently, $\Lambda(a, b^2) = \Lambda(a, b) = S_0$.

Explicit results on $\Lambda(a, b)$ are given in the following proposition.

Proposition 13 Let $b \in \mathbb{F}_{16}^*$, then

- (1) If $b = 1$, $\Lambda(a, b) = S_0 - 2(S_1 + S_2) = 2S_0 - \Lambda(a, 0)$.
- (2) If $b \in \{\beta + \beta^2, \beta + \beta^3, \beta^2 + \beta^4, \beta^3 + \beta^4\}$, that is, b is a primitive element satisfies $\text{Tr}_1^4(b) = 0$, then $\Lambda(a, b) = S_0$.
- (3) If $b = \beta$ or β^4 , then $\Lambda(a, b) = -S_0 - 2S_1$.
- (4) If $b = \beta^2$ or β^3 , then $\Lambda(a, b) = -S_0 - 2S_2$.
- (5) If $b = 1 + \beta$ or $1 + \beta^4$, then $\Lambda(a, b) = -S_0 + 2S_1$.
- (6) If $b = 1 + \beta^2$ or $1 + \beta^3$, $\Lambda(a, b) = -S_0 + 2S_2$.
- (7) If $b = \beta + \beta^4$, then $\Lambda(a, b) = S_0 + 2S_1 - 2S_2$.
- (8) If $b = \beta^2 + \beta^3$, then $\Lambda(a, b) = S_0 - 2S_1 + 2S_2$.

Proof From Proposition 10, this proposition follows.

Further, if $a \in \mathbb{F}_{2^{m_1}}$, we have the following proposition.

Proposition 14 *If $a \in \mathbb{F}_{2^{m_1}}$, then*

- (1) *If $b = 1$, then $\Lambda(a, b) = 2S_0 - \Lambda(a, 0)$.*
- (2) *If $b \in \{\beta, \beta^2, \beta^3, \beta^4\}$, then $\Lambda(a, b) = -S_0 - 2S_1 = -\frac{S_0 + \Lambda(a, 0)}{2}$.*
- (3) *If $b \in \{1 + \beta, 1 + \beta^2, 1 + \beta^3, 1 + \beta^4\}$, then $\Lambda(a, b) = -S_0 + 2S_1 = -\frac{3S_0 - \Lambda(a, 0)}{2}$.*
- (4) *If $b \in \{\beta + \beta^2, \beta + \beta^3, \beta^2 + \beta^4, \beta^3 + \beta^4, \beta + \beta^4, \beta^2 + \beta^3\}$, then $\Lambda(a, b) = S_0$.*

Proof From Proposition 11 and Proposition 13, this proposition follows.

Corollary 4 *For $a \in \mathbb{F}_{2^{m_1}}$, $\Lambda(a, b^2) = \Lambda(a, b)$.*

Proof This corollary can be obtained by Proposition 14.

We now introduce some results on character sums.

Lemma 2 *Let U be the group of $2^m + 1$ -th roots of unity in $\mathbb{F}_{2^n}^*$, then for any positive integer p ,*

$$\sum_{u \in U} \chi(\text{Tr}(ax^{p(2^m-1)})) = 1 + 2 \sum_{x \in \mathbb{F}_{2^m}^*, \text{Tr}_1^m(x^{-1})=1} \chi(\text{Tr}(aD_p(x))).$$

Proof This lemma is a special case of Lemma 12 by Mesnager [32].

Proposition 15 *Let S_0 and $\Lambda(a, 0)$ be defined above, then*

- (1) $\Lambda(a, 0) = 1 - K_m(a)$.
- (2) $S_0 = \frac{1}{5}[1 - K_m(a) + 2Q_m(a)]$.

Proof (1) From Lemma 2,

$$\begin{aligned} \Lambda(a, 0) &= \sum_{u \in U} \chi(\text{Tr}_1^n(au^{2^m-1})) \\ &= 1 + 2 \sum_{x \in \mathbb{F}_{2^m}^*, \text{Tr}_1^m(x^{-1})=1} \chi(\text{Tr}_1^m(ax)) \\ &= 1 + 2 \cdot \frac{1}{2} \left[\sum_{x \in \mathbb{F}_{2^m}} \chi(\text{Tr}_1^m(ax)) - \sum_{x \in \mathbb{F}_{2^m}} \chi(\text{Tr}_1^m(ax + \frac{1}{x})) \right]. \end{aligned}$$

Note that $\sum_{x \in \mathbb{F}_{2^m}} \chi(\text{Tr}_1^m(ax)) = 0$, then

$$\Lambda(a, 0) = 1 - \sum_{x \in \mathbb{F}_{2^m}} \chi(\text{Tr}_1^m(ax + \frac{1}{x})) = 1 - K_m(a).$$

(2) Note that $S_0 = \sum_{v \in V} \chi(\text{Tr}_1^n(av^{2^m-1})) = \frac{1}{5} \sum_{u \in U} \chi(\text{Tr}_1^n(au^{5(2^m-1)}))$. Then from Lemma 2,

$$\begin{aligned} S_0 &= \frac{1}{5} [1 + 2 \sum_{x \in \mathbb{F}_{2^m}^*, \text{Tr}_1^m(x^{-1})=1} \chi(\text{Tr}_1^m(aD_5(x)))] \\ &= \frac{1}{5} [1 + 2 \sum_{x \in \mathbb{F}_{2^m}} \chi(\text{Tr}_1^m(aD_5(x))) - 2 \sum_{x \in \mathbb{F}_{2^m}, \text{Tr}_1^m(x^{-1})=0} \chi(\text{Tr}_1^m(aD_5(x)))]. \end{aligned}$$

Since $\frac{1}{D_5(x)} = \frac{1}{x^5+x^3+x} = \frac{1}{x} + \frac{x}{x^2+x+1} + (\frac{x}{x^2+x+1})^2$, $\text{Tr}_1^m(\frac{1}{D_5(x)}) = \text{Tr}_1^m(\frac{1}{x})$. Then $D_5(x)$ induces the map

$$\begin{aligned} \{x \in \mathbb{F}_{2^m} | \text{Tr}_1^m(x^{-1}) = 0\} &\longrightarrow \{x \in \mathbb{F}_{2^m} | \text{Tr}_1^m(x^{-1}) = 0\} \\ x &\longmapsto D_5(x). \end{aligned}$$

From Proposition 4, this map is a permutation of $\{x \in \mathbb{F}_{2^m} | \text{Tr}_1^m(x^{-1}) = 0\}$. Hence,

$$\begin{aligned} &\sum_{x \in \mathbb{F}_{2^m}, \text{Tr}_1^m(x^{-1})=0} \chi(\text{Tr}_1^m(aD_5(x))) \\ &= \sum_{x \in \mathbb{F}_{2^m}, \text{Tr}_1^m(x^{-1})=0} \chi(\text{Tr}_1^m(ax)) \\ &= \frac{1}{2} \left[\sum_{x \in \mathbb{F}_{2^m}} \chi(\text{Tr}_1^m(ax)) + \sum_{x \in \mathbb{F}_{2^m}} \chi(\text{Tr}_1^m(ax + \frac{1}{x})) \right] \\ &= \frac{1}{2} K_m(a). \end{aligned}$$

Consequently, $S_0 = \frac{1}{5} [1 + 2Q_m(a) - K_m(a)]$.

The relation between $K_m(a)$, $Q_m(a)$ and $\Lambda(a, b)$ is given in the following proposition.

Proposition 16 *Let $\Lambda(a, b)$ be defined above, where $a \in \mathbb{F}_{2^m}$, $b \in \mathbb{F}_{16}^*$, then*

(1) *If $b = 1$, then $\Lambda(a, 1) = -\frac{1}{5} [3(1 - K_m(a)) - 4Q_m(a)]$.*

(2) *If b is a primitive element such that $\text{Tr}_1^4(b) = 0$, then $\Lambda(a, b) = \frac{1}{5} [1 - K_m(a) + 2Q_m(a)]$.*

Proof From Proposition 13, $\Lambda(a, 1) = 2S_0 - \Lambda(a, 0)$. Further, Proposition 15,

$$\begin{aligned} \Lambda(a, 1) &= 2 \cdot \frac{1}{5} [1 - K_m(a) + 2Q_m(a)] - (1 - K_m(a)) \\ &= -\frac{3}{5} (1 - K_m(a)) + \frac{4}{5} Q_m(a) \\ &= -\frac{1}{5} [3(1 - K_m(a)) - 4Q_m(a)] \end{aligned}$$

(2) From Proposition 13 and Proposition 15, $\Lambda(a, b) = S_0 = \frac{1}{5} [1 - K_m(a) + 2Q_m(a)]$.

3.3 The hyper-bentness of $f_{a,b}(a \in \mathbb{F}_{2^m}, (b+1)(b^4+b+1)=0)$

Proposition 17 *Let $n = 2m$, $m = 2m_1$, $m_1 \equiv 1 \pmod{2}$ and $m_1 \geq 3$, the Boolean function $f_{a,1}$ in \mathcal{H}_n of the form*

$$\text{Tr}_1^n(ax^{2^m-1}) + \text{Tr}_1^4(x^{\frac{2^n-1}{5}})$$

is a hyper-bent function if and only if $Q_m(a) = 2^{m_1}$ and $K_m(a) = \frac{4}{3}(2 - 2^{m_1})$ holds.

Proof From Proposition 9, $f_{a,1}$ is a hyper-bent function if and only if $\Lambda(a, 1) = 1$. From Proposition 16, $\Lambda(a, 1) = -\frac{1}{5}[3(1 - K_m(a)) - 4Q_m(a)]$. Then $\Lambda(a, 1) = 1$ if and only if $K_m(a) = \frac{4}{3}(2 - Q_m(a))$. Note that

$$1 - 2 \cdot 2^{m_1} \leq K_m(a) \leq 1 + 2 \cdot 2^{m_1}.$$

Further, we have

$$-\frac{3}{2} \cdot 2^{m_1} + \frac{5}{4} \leq Q_m(a) \leq \frac{3}{2} \cdot 2^{m_1} + \frac{5}{4}.$$

From $m_1 \geq 3$,

$$-2 \cdot 2^{m_1} \leq Q_m(a) \leq 2 \cdot 2^{m_1}.$$

From Corollary 1, $Q_m(a) \in \{0, \pm 2^{m_1}, \pm 2 \cdot 2^{m_1}, \pm 4 \cdot 2^{m_1}\}$. Hence, the value of $Q_m(a)$ is 0 or $\pm 2^{m_1}$.

If $Q_m(a) = 0$, then $K_m(a) = \frac{4}{3}(2 - 0) = \frac{8}{3}$. Since $K_m(a)$ is an integer, $Q_m(a) \neq 0$.

If $Q_m(a) = -2^{m_1}$, then $K_m(a) = \frac{4}{3}(2 + 2^{m_1}) = \frac{8}{3}(1 + 2^{m_1-1})$. Since m_1 is odd, $3 \nmid (1 + 2^{m_1-1})$. Hence, $Q_m(a) \neq -2^{m_1}$.

As a result, $f_{a,1}$ is a hyper-bent function if and only if $Q_m(a) = 2^{m_1}$. Then $K_m(a) = \frac{4}{3}(2 - 2^{m_1})$. Hence, this proposition follows.

Proposition 18 *Let $n = 2m$, $m = 2m_1$, $m_1 \equiv 1 \pmod{2}$ and $m_1 \geq 3$. Let b be a primitive element in \mathbb{F}_{16}^* such that $\text{Tr}_1^4(b) = 0$. The Boolean function $f_{a,b}$ in \mathcal{H}_n of the form*

$$\text{Tr}_1^n(ax^{2^m-1}) + \text{Tr}_1^4(bx^{\frac{2^n-1}{5}})$$

is a hyper-bent function if and only if one of the following assertions (1) and (2) holds.

- (1) $Q_m(a) = 0$, $K_m(a) = -4$.
- (2) $Q_m(a) = 2^{m_1}$, $K_m(a) = 2 \cdot 2^{m_1} - 4$.

Proof $f_{a,b}$ is a hyper-bent function if and only if $\Lambda(a, b) = 1$. From Proposition 13, when b is a primitive element such that $\text{Tr}_1^4(b) = 0$, $\Lambda(a, b) = S_0$. From Proposition 15, $\Lambda(a, b) = \frac{1}{5}[1 - K_m(a) + 2Q_m(a)]$. Hence, $\Lambda(a, b) = 1$ if and only if $K_m(a) = 2Q_m(a) - 4$.

From Corollary 1, $Q_m(a) \in \{0, \pm 2^{m_1}, \pm 2 \cdot 2^{m_1}, \pm 4 \cdot 2^{m_1}\}$. Further, from Proposition 6, $K_m(a) \in [1 - 2 \cdot 2^{m_1}, 1 + 2 \cdot 2^{m_1}]$. Hence, $Q_m(a) = 0$ or 2^{m_1} . If $Q_m(a) = 0$, then $K_m(a) = -4$. If $Q_m(a) = 2^{m_1}$, then $K_m(a) = 2 \cdot 2^{m_1} - 4$. Therefore, this proposition follows.

Theorem 2 Let $n = 2m$, $m = 2m_1$, $m_1 \equiv 1 \pmod{2}$ and $m_1 \geq 3$. The Boolean function $f_{a,1}$ in \mathcal{H}_n of the form

$$\text{Tr}_1^n(ax^{2^m-1}) + \text{Tr}_1^4(x^{\frac{2^n-1}{5}})$$

is a hyper-bent function if and only if the following assertions holds.

- (1) $p(x) = x^5 + x + a^{-1}$ is irreducible over \mathbb{F}_{2^m} .
- (2) The quadratic form $\mathbf{q}(x) = \text{Tr}_1^m(x(ax^4 + ax^2 + a^2x))$ over \mathbb{F}_{2^m} is even.
- (3) $K_m(a) = \frac{4}{3}(2 - 2^{m_1})$.

Proof From Proposition 17 and Corollary 2, this theorem follows.

Theorem 3 Let $n = 2m$, $m = 2m_1$, $m_1 \equiv 1 \pmod{2}$ and $m_1 \geq 3$. Let b be a primitive element in \mathbb{F}_{16}^* such that $\text{Tr}_1^4(b) = 0$. The Boolean function $f_{a,b}$ in \mathcal{H}_n of the form

$$\text{Tr}_1^n(ax^{2^m-1}) + \text{Tr}_1^4(bx^{\frac{2^n-1}{5}})$$

is a hyper-bent function if and only if one of the assertions (1) and (2) holds.

- (1) $p(x) = x^5 + x + a^{-1}$ over \mathbb{F}_{2^m} is (1)(2)² and $K_m(a) = -4$.
- (2) $p(x) = x^5 + x + a^{-1}$ is irreducible over \mathbb{F}_{2^m} . The quadratic form $\mathbf{q}(x) = \text{Tr}_1^m(x(ax^4 + ax^2 + a^2x))$ over \mathbb{F}_{2^m} is even. $K_m(a) = 2 \cdot 2^{m_1} - 4$.

Proof From 18 and Corollary 2, this theorem follows.

3.4 The hyper-bentness of $f_{a,b}$ ($a \in \mathbb{F}_{2^{\frac{m}{2}}}, b \in \mathbb{F}_{16}$)

To consider the hyper-bentness of $f_{a,b}$ ($a \in \mathbb{F}_{2^{\frac{m}{2}}}, b \in \mathbb{F}_{16}$), we require more properties of $K_m(a)$ and $Q_m(a)$.

Lemma 3 Let $a \in \mathbb{F}_{2^{m_1}}^*$, $m = 2m_1$ and $p(x) = x^5 + x + a^{-1}$, then

- (1) $1 - K_m(a) = (1 - K_{m_1}(a))^2 - 2 \cdot 2^{m_1}$.
- (2) If $m_1 \equiv 1 \pmod{2}$, then $Q_m(a) \in \{0, 2 \cdot 2^{m_1}, -4 \cdot 2^{m_1}\}$. Further, $Q_m(a) = 0$ if and only if $p(x) = (1)(4)$. $Q_m(a) = 2 \cdot 2^{m_1}$ if and only if $p(x) = (2)(3)$. $Q_m(a) = -4 \cdot 2^{m_1}$ if and only if $p(x) = (1)^3(2)$.

Proof (1) We introduce an auxiliary curve

$$C : y^2 + y = ax + \frac{1}{x}, a \in \mathbb{F}_{2^{m_1}},$$

that is, $xy^2 + xy = ax^2 + 1$. Hence, C is an elliptic curve. There are two infinite points on C and the x coordinate of any point on C is not zero. Therefore,

$$\begin{aligned}
\#(C(\mathbb{F}_{2^{m_1}})) &= 2 + 2 \cdot \#\{x \in \mathbb{F}_{2^{m_1}}^* \mid \text{Tr}_1^{m_1}(ax + \frac{1}{x}) = 0\} \\
&= 2 \cdot \#\{x \in \mathbb{F}_{2^{m_1}} \mid \text{Tr}_1^{m_1}(ax + \frac{1}{x}) = 0\} \\
&= \cdot \#\{x \in \mathbb{F}_{2^{m_1}} \mid \text{Tr}_1^{m_1}(ax + \frac{1}{x}) = 0\} - 2^{m_1} + 2^{m_1} \\
&= \sum_{x \in \mathbb{F}_{2^{m_1}}} \chi(\text{Tr}_1^{m_1}(ax + \frac{1}{x})) + 2^{m_1} \\
&= (1 + 2^{m_1}) + K_{m_1}(a) - 1.
\end{aligned}$$

Further, $\#(C(\mathbb{F}_{2^{2m_1}})) = (1 + 2^{2m_1}) + K_{2m_1}(a) - 1$. From properties of elliptic curves, we have

$$1 - K_m(a) = 1 - K_{2m_1}(a) = (1 - K_{m_1}(a))^2 - 2 \cdot 2^{m_1}.$$

(2) Consider $p(x) = x^5 + x + a^{-1}$ over $\mathbb{F}_{2^{m_1}}$. From Proposition 8 and (1),

$$Q_m(a) = 2s - r^2,$$

where (r, s) is determined by m_1 and Proposition 7. Therefore, $Q_m(a) = 0$ if and only if $p(x) = (1)(4)$. $Q_m(a) = 2 \cdot 2^{m_1}$ if and only if $p(x) = (2)(3)$. $Q_m(a) = -4 \cdot 2^{m_1}$ if and only if $p(x) = (1)^3(2)$.

Corollary 5 *Let $a \in \mathbb{F}_{2^{m_1}}$, $m = 2m_1$ and $m_1 \equiv 1 \pmod{2}$, then $K_m(a) \neq -4$.*

Proof From Lemma 3, if $K_m(a) = -4$,

$$(1 - K_{m_1}(a))^2 = 2 \cdot 2^{m_1} + 5. \quad (5)$$

Since $m_1 \geq 3$ and $m_1 \equiv 1 \pmod{2}$, $(2^{\frac{m_1+1}{2}})^2 < (1 - K_{m_1}(a))^2 < (2^{\frac{m_1+1}{2}} + 1)^2$. Then (5) has no integer solution. Hence $K_m(a) \neq -4$.

Theorem 4 *Let $n = 2m$, $m = 2m_1$, $m_1 \equiv 1 \pmod{2}$ and $m_1 \geq 3$. If $b \in \mathbb{F}_{16} \setminus \{\beta^i \mid 1 \leq i \leq 4\}$, then the Boolean function $f_{a,b}$ in \mathcal{H}_n of the form*

$$\text{Tr}_1^n(ax^{2^m-1}) + \text{Tr}_1^4(bx^{\frac{2^n-1}{5}}), a \in \mathbb{F}_{2^{m_1}}$$

is not a hyper-bent function.

Proof When $b = 0$, $f_{a,0}$ is a hyper-bent function if and only if $\Lambda(a, 0) = 1$. From Proposition 15, $\Lambda_{a,0} = 1 - K_m(a)$. Hence, $f_{a,0}$ is a hyper-bent function if and only if $K_m(a) = 0$. From Lemma 3, $(1 - K_m(a))^2 = 2 \cdot 2^{m_1} + 1$. m_1 is odd, then

$$(2^{\frac{m_1+1}{2}})^2 < 2 \cdot 2^{m_1} + 1 < (2^{\frac{m_1+1}{2}} + 1)^2.$$

$2 \cdot 2^{m_1} + 1$ is not a square. Then $f_{a,0}$ is not a hyper-bent function.

When $b = 1$, from Proposition 14, $\Lambda(a, 1) = 2S_0 - \Lambda(a, 0)$. From Proposition 15,

$$\Lambda(a, 1) = -\frac{3}{5}(1 - K_m(a)) + \frac{4}{5}Q_m(a) = -\frac{1}{5}[3(1 - K_m(a)) - 4Q_m(a)].$$

From Proposition 9, $f_{a,1}$ is a hyper-bent function if and only if $3(1 - K_m(a)) - 4Q_m(a) = -5$. From Lemma 3,

$$3(1 - K_{m_1}(a))^2 = 6 \cdot 2^{m_1} + 4Q_m(a) - 5. \quad (6)$$

Note that $Q_m(a) \in \{0, 2 \cdot 2^{m_1}, -4 \cdot 2^{m_1}\}$. If $Q_m(a) = 0$, (6) does not hold. If $Q_m(a) = 2 \cdot 2^{m_1}$, then

$$3(1 - K_{m_1}(a))^2 = 14 \cdot 2^{m_1} - 5. \quad (7)$$

From Proposition 6, $1 - K_{m_1}(a) \in [-2 \cdot 2^{m_1}, 2 \cdot 2^{m_1}]$. Since $m_1 \geq 3$, (7) does not hold. If $Q_m(a) = -4 \cdot 2^{m_1}$, then

$$3(1 - K_{m_1}(a))^2 = -10 \cdot 2^{m_1} - 5, \quad (8)$$

Obviously, (8) does not hold. As a result, $f_{a,1}$ is not a hyper-bent function.

When $b \in \{\beta + \beta^2, \beta + \beta^3, \beta^2 + \beta^4, \beta^3 + \beta^4, \beta + \beta^4, \beta^2 + \beta^3\}$, $\Lambda(a, b) = S_0$. From Proposition 15, $\Lambda(a, b) = \frac{1}{5}[1 - K_m(a) + 2Q_m(a)]$. If $f_{a,b}$ is a hyper-bent function, then $\Lambda(a, b) = 1$. Hence, $1 - K_m(a) = 5 - 2Q_m(a)$. Note that $Q_m(a) \in \{0, 2 \cdot 2^{m_1}, -4 \cdot 2^{m_1}\}$. Since $1 - K_m(a) \in [-2 \cdot 2^{m_1}, 2 \cdot 2^{m_1}]$, $Q_m(a) = 0$ and $1 - K_m(a) = 5$. Then $K_m(a) = -4$. From Corollary 5, $K_m(a) \neq -4$, which gives a contradiction. Hence, $f_{a,b}$ is not a hyper-bent function.

When $b \in \{1 + \beta, 1 + \beta^2, 1 + \beta^3, 1 + \beta^4\}$, $\Lambda(a, b) = -\frac{3S_0 - \Lambda(a, 0)}{2}$. From Proposition 15, $\Lambda(a, b) = \frac{1}{5}[1 - K_m(a) - 3Q_m(a)]$. If $f_{a,b}$ is a hyper-bent function, $1 - K_m(a) = 3Q_m(a) + 5$. Further, we have $Q_m(a) = 0$ and $1 - K_m(a) = 5$, that is, $K_m(a) = -4$, which contradicts Corollary 5. Hence, $f_{a,b}$ is not a hyper-bent function.

Above all, this theorem follows.

We now introduce some results on generalized Ramanujan-Nagell equations [1, 22, 23]. A generalized Ramanujan-Nagell equation is of the form

$$D_1x^2 + D_2 = \eta^2 \cdot p^k,$$

where D_1, D_2 are two positive integers, p is a prime and $\eta \in \{1, \sqrt{2}, 2\}$.

Generally, a generalized Ramanujan-Nagell equation have no more that a solution. Some exceptions are listed in [1]. In particular, we have the following lemma.

Lemma 4 *The equation $15x^2 + 1 = 2 \cdot 2^k$ has only a solution $(x, k) = (1, 3)$. The equation $3x^2 + 5 = 4 \cdot 2^k$ has three solutions $\{(x, k) | (1, 1), (3, 3), (13, 7)\}$.*

Theorem 5 Let $n = 2m$, $m = 2m_1$, $m_1 \equiv 1 \pmod{2}$ and $m_1 \geq 3$. If $n \neq 12, 28$, then the Boolean functions $f_{a,b}$ in \mathcal{H}_n of the form

$$\text{Tr}_1^n(ax^{2^m-1}) + \text{Tr}_1^4(bx^{\frac{2^n-1}{5}}), a \in \mathbb{F}_{2^{m_1}}, b \in \mathbb{F}_{16},$$

is not a hyper-bent function. Further, if $n = 12$, all the hyper-bent functions with $a \in \mathbb{F}_{2^3}$ in \mathcal{H}_{12} are $\text{Tr}_1^{12}(ax^{2^6-1}) + \text{Tr}_1^4(bx^{\frac{2^{12}-1}{5}})$, where $(a+1)(a^3+a^2+1) = 0$ and $b = \beta^i, i = 1, 2, 3, 4$. If $n = 28$, all the hyper-bent functions with $a \in \mathbb{F}_{2^7}$ in \mathcal{H}_{28} are $\text{Tr}_1^{28}(ax^{2^{14}-1}) + \text{Tr}_1^4(bx^{\frac{2^{28}-1}{5}})$, where $(a+1)(a^7+a^6+a^5+a^4+a^3+a^2+1) = 0$ and $b = \beta^i, i = 1, 2, 3, 4$.

Proof If $b = \beta^i$ ($i = 1, 2, 3, 4$), then $\Lambda(a, b) = -\frac{S_0 + \Lambda(a, 0)}{2}$. From Proposition 15,

$$\Lambda(a, b) = -\frac{1}{5}[3(1 - K_m(a)) + Q_m(a)].$$

Then $f_{a,b}$ is a hyper-bent function if and only if $\Lambda(a, b) = 1$, that is,

$$3(1 - K_m(a)) + Q_m(a) = -5. \quad (9)$$

From Lemma 3, $Q_m(a) \in \{0, 2 \cdot 2^{m_1}, -4 \cdot 2^{m_1}\}$. If $Q_m(a) = 0$, (9) does not hold.

From Lemma 3,

$$3(1 - K_{m_1}(a))^2 = 6 \cdot 2^{m_1} - Q_m(a) - 5. \quad (10)$$

If $Q_m(a) = 2 \cdot 2^{m_1}$,

$$3(1 - K_m(a))^2 + 5 = 4 \cdot 2^{m_1}. \quad (11)$$

From Lemma 4, we have $(K_{m_1}(a), m_1) \in \{(4, 3), (-2, 3), (14, 7), (-12, 7)\}$. Since $4 \nmid K_{m_1}(a)$, $(K_{m_1}(a), m_1) = (4, 3)$ or $(-12, 7)$.

If $Q_m(a) = -4 \cdot 2^{m_1}$, then $3(1 - K_{m_1}(a))^2 = 5(2 \cdot 2^{m_1} - 1)$. Hence, $5 \mid 1 - K_{m_1}(a)$. Then

$$2 \cdot 2^{m_1} = 15\left(\frac{1 - K_{m_1}(a)}{5}\right)^2 + 1. \quad (12)$$

From Lemma 4, $(K_{m_1}(a), m_1) = (6, 3)$ or $(-4, 3)$. Since $4 \nmid K_{m_1}(a)$, $(K_{m_1}(a), m_1) = (-4, 3)$.

If $(K_{m_1}(a), m_1) = (4, 3)$, then $Q_m(a) = 2 \cdot 2^{m_1}$. If $(K_{m_1}(a), m_1) = (-4, 3)$, then $Q_m(a) = -4 \cdot 2^{m_1}$. From Lemma 3, if $m_1 = 3$, then $n = 12$ and f_{a,β^i} ($a \in \mathbb{F}_8, i = 1, 2, 3, 4$) is a hyper-bent function if and only if one of the assertions (1) and (2) holds.

(1) $p(x) = x^5 + x + a^{-1} = (2)(3)$ and $K_3(a) = 4$.

(2) $p(x) = x^5 + x + a^{-1} = (1)^3(2)$ and $K_3(a) = -4$.

If $(K_{m_1}(a), m_1) = (-12, 7)$, then $Q_m(a) = 2 \cdot 2^{m_1}$. From Lemma 3, if $m_1 = 7$, then $n = 28$ and f_{a,β^i} ($a \in \mathbb{F}_{128}, i = 1, 2, 3, 4$) is a hyper-bent function if and only if $p(x) = x^5 + x + a^{-1} = (2)(3)$ and $K_7(a) = -12$.

With the help of experiments on the computer, if $m_1 = 3$, then $n = 12$ and all the hyper-bent functions with $a \in \mathbb{F}_8$ in \mathcal{H}_{12} are

$$\text{Tr}_1^{12}(ax^{2^6-1}) + \text{Tr}_1^4(bx^{\frac{2^{12}-1}{5}}),$$

where $(a+1)(a^3+a^2+1)=0$ and $b=\beta^i$ ($i=1,2,3,4$).

If $m_1 = 7$, then $n = 28$ and all the hyper-bent functions with $a \in \mathbb{F}_{128}$ in \mathcal{H}_{28} are

$$\text{Tr}_1^{28}(ax^{2^{14}-1}) + \text{Tr}_1^4(bx^{\frac{2^{28}-1}{5}}),$$

where $(a+1)(a^7+a^6+a^5+a^4+a^3+a^2+1)=0$ and $b=\beta^i$ ($i=1,2,3,4$).

Above all, this theorem follows.

4 Conclusion

This paper considers the hyper-bentness of the Boolean functions $f_{a,b}$ of the form $f_{a,b} := \text{Tr}_1^n(ax^{2^m-1}) + \text{Tr}_1^4(bx^{\frac{2^n-1}{5}})$, where $n = 2m$, $m \equiv 2 \pmod{4}$, $a \in \mathbb{F}_{2^m}$ and $b \in \mathbb{F}_{16}$. If $b = 1$ or b is a primitive element in \mathbb{F}_{16} such that $\text{Tr}_1^4(b) = 0$, the hyper-bentness of $f_{a,b}$ can be characterized by Kloosterman sums and the factorization of $x^5 + x + a^{-1}$. If $a \in \mathbb{F}_{2^{\frac{m}{2}}}^*$, with the help of generalized Ramanujan-Nagell equations, we prove that $f_{a,b}$ is not a hyper-bent function unless $n = 12$ or $n = 28$. Further, we give all the hyper-bent functions for $n = 12$ or $n = 28$.

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